MODULE - VI

18

FITTING OF STRAIGHT LINE

Consider the fitting of straight line $\boxed{Y = a + bX}$ to the data (x_i, y_i) $i = 1, 2, \ldots$ Here a is the y intercept and b is the slope of the line. To determine the values of a and b, we solve the normal eqⁿs,

$$\boxed{\begin{aligned} \Sigma Y &= na + b\Sigma X \\ \Sigma XY &= a\Sigma X + b\Sigma X^2 \end{aligned}}$$

Where n represents the no. of ordered pairs.

Fit a straight line to the following data.

x :	1	2	3	4	5
y :	14	27	40	55	68

X	Y	XY	X^2
1	14	14	1
2	27	54	4
3	40	120	9
4	55	220	16
5	68	340	25
= 15	$\Sigma Y = 204$	$\Sigma XY = 748$	$\Sigma X^2 = 55$

The normal eqⁿs are,

$$\Sigma Y = na + b\Sigma X$$

$$\Sigma XY = a\Sigma X + b\Sigma X^2$$

$$\Sigma Y = 204 = 5a + b \times 15 \quad -(1)$$

$$748 = 15a + 55b \quad -(2)$$

solving ;

$a = 0$ $b = 13.6$

$\therefore$ The eqn of the straight line is $y = a + bX$

ie, $y = 0 + 13.6X$

$y = 13.6X$

8 FITTING OF PARABOLA

Consider the fitting of the parabola of the form $\boxed{y = a + bx + cx^2}$ to the data (x_i, y_i) $i = 1, 2, \ldots, n$, where 'a' is the y intercept, 'b' is the slope of the curve at the origin and 'c' is

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the rate of change in the slope. To determine the value a, b, c, we solve the normal eqn's,

$$\boxed{\begin{aligned} \Sigma y &= na + b\Sigma x + c\Sigma x^2 \\ \Sigma xy &= a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \\ \Sigma x^2 y &= a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \end{aligned}}$$

Where n represents the no of ordered pairs

? Fit a parabola to the following data.

x	1	2	3	4	5
y	10	12	8	10	14

Ans.

x	y	x^2	x^3	x^4	xy	x^2y
1	10	1	1	1	10	10
2	12	4	8	16	24	48
3	8	9	27	81	24	72
4	10	16	64	256	40	160
5	14	25	125	625	70	350
$\Sigma x = 15$	$\Sigma y = 54$	$\Sigma x^2 = 55$	$\Sigma x^3 = 225$	$\Sigma x^4 = 979$	$\Sigma xy =$	$\Sigma x^2y = 640$

∴ The normal eqn are,

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

$$54 = 5a + 15b + 55c \quad \text{--- (1)}$$

$$168 = 15a + 55b + 225c \quad \text{--- (2)}$$

$$640 = 55a + 225b + 979c \quad — (3)$$

Solving,

$$a = 14, \quad b = -3.68 \quad c = 0.71$$

$\therefore$ The eqn of the parabola is,

$$Y = a + bx + cx^2$$

$$Y = 14 - 3.6x + 0.71x^2$$

GAUSS - ELIMINATION METHOD

In this method, we can solve 'n' eq^ns in n unknowns by reducing the system of equations to an upper triangular system which can be solved by back substitution method

Consider 'n' eq^ns in 'n' unknowns as,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

The above system can be written as.

$$AX = B$$

ie.
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Consider the augmented matrix
$$[A \,\vdots\, B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

Now we have to reduce the augmented matrix into an upper triangular matrix by using elementary row operations. Then the solution is obtained by using back substitution.

Q. Solve by Gauss-elimination method.

$$x + y + z = 9$$

$$2x - 3y + 4z = 13$$

$$3x + 4y + 5z = 40$$

The given system can be written in the form.

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Augmented matrix $= \left[A \,\vdots\, B \right]$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & -5 & 2 & -5 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{cccc} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 12 & 60 \end{array}\right] \qquad R_3 \to R_3 + 5R_2$$

$12z = 60$

$z = 60/12$

$z = 5$ //

$y + 2z = 13$

$\underline{\underline{y = 3}}$

$x + y + z = 9$

$x + 3 + 5 = 9$

$\underline{x = 1}$

$\therefore$ The solution is,

$x = 1, y = 3,$

NUMERICAL INTEGRATION

TRAPEZOIDAL RULE

$$\int_a^b F(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

SIMPSON'S $\frac{1}{3}$ RD RULE

$$\int_a^b F(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\cdots)+2(y_2+y_4+\cdots)\right]$$

SIMPSON'S $\frac{3}{8}$ RULE

$$\int_a^b F(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+\cdots)+2(y_3+y_6+\cdots)\right]$$

Where $h = \frac{b-a}{n}$

NOTE

Simpson's $\frac{3}{8}$ rule can be applied if the no. of intervals is a multiple of 3.

? Evaluate $\int_{-3}^{3} x^4\,dx$ by using.

(i) Trapezoidal

(ii) Simpson's rule.

Varify the results by actual integration.

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Trapezoidal rule

$$\int_a^b F(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

$$\int_{-3}^{3} x^4\,dx = \frac{1}{2}\left[(81 + 81) + 2(16 + 1 + 0 + 1 + 16)\right]$$

$$= 115$$

Simpson's $\frac{1}{3}$ rd rule

$$\int_a^b F(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots)\right]$$

$$\int_{-3}^{3} x^4\,dx = \frac{1}{3}\left[(81 + 81) + 4(16 + 0 + 16) + 2(1+1)\right]$$

$$= 98$$

Simpson's $\frac{3}{8}$ th rule

$$\int_a^b F(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \cdots) + 2(y_3 + y_6 + \cdots)\right]$$

$$\int_{-3}^{3} x^4\,dx = \frac{3\times 1}{8}\left[(81 + 81) + 3(16 + 1 + 1 + 16) + 2(0)\right]$$

$$= 99$$

By actual Integration,

$$\int_{-3}^{3} x^4\,dx = \left(\frac{x^5}{5}\right)_{-3}^{3} = \frac{1}{5}\left(3^5 - (-3)^5\right)$$

$$= 97.2$$

(iii) Simpson's $\frac{3}{8}$ rule.

GAUSS QUADRATURE METHOD

GAUSS TWO POINT QUADRATURE FORMULA

Consider the integral $I = \int_{-1}^{1} F(x)\,dx$. Then we have,

$$\boxed{I = \int_{-1}^{1} F(x)\,dx = F(0.5773) + F(-0.5773)}$$

This is known as Gauss two point quadrature formula.

NOTE

In Gauss two point quadrature formula, the limit of integration is from -1 to 1. If the limit is from a to b, then we shall applied a suitable change of variable to print the limits from -1 to 1. We replace the given variable x by another variable 't' which are related by the formula,

$$\boxed{x = \frac{(b-a)t + (b+a)}{2}} \qquad \text{then } dx = \left(\frac{b-a}{2}\right) dt.$$

$$\boxed{\therefore \int_a^b F(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} F\left[\frac{(b-a)t + (b+a)}{2}\right] dt}$$

GAUSS THREE POINT QUADRATURE FORMULA

$$\boxed{I = \int_{-1}^{1} F(x)\,dx = 0.55555\,F(-0.77459) + 0.88888\,F(0) + 0.55555 \times F(0.77459)}$$

Evaluate $\int_{-1}^{1} e^{-x^2}\,dx$ by using Gauss two point and three point formula.

Here $a=-1,\ b=1,\ F(x)=e^{-x^2}$.

By Gauss two point quadriature formula,

$$\int_{-1}^{1} F(x)\,dx = F(0.57773) + F(-0.57773)$$

$$= \cancel{0.7165 + 1.3955}\ \underline{\underline{1.433}}$$

By Gauss three point quadrature formula,

$$I = \underline{\underline{1.4986}}$$

PARTIAL DIFFERENTIAL EQUATIONS

A partial differential eqⁿ (PDE) is an eqⁿ that involves an unknown function and sum of its partial derivatives w.r.t two or more independent variables.

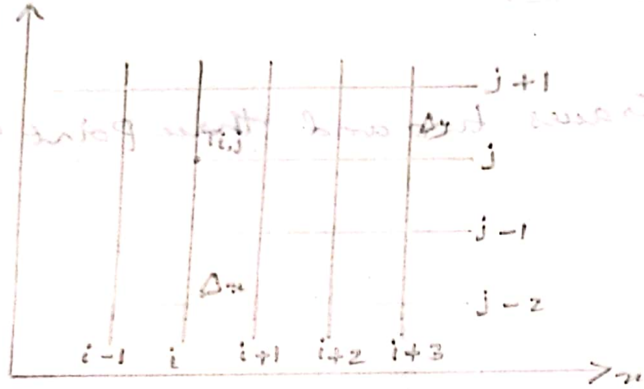
Eg:- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$$

For solving APD, two different types of additional conditions must be provided in order to select the desired solution from many possible solutions. These additional conditions are called initial conditions and boundary conditions.

Initial conditions are conditions to be satisfied everywhere for a fixed value of an independent variable. Boundary conditions are conditions to be satisfied everywhere on the boundary of the region on which the PDE is defined.

FINITE DIFFERENCE METHOD

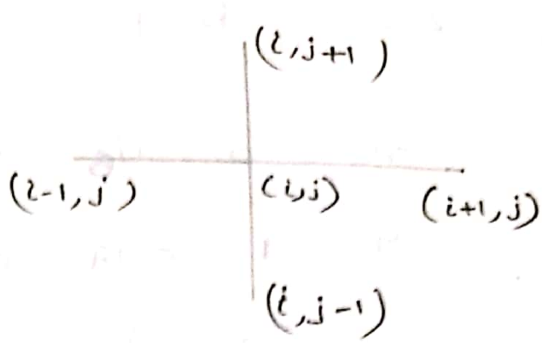


Consider a two dimensional region given above. The two sets of lines drawn parallel to the co-ordinate axes form a rectangular grid having spacing Δx and Δy along the x and y directions. The points of intersection of these lines are called mesh points or nodal points.

Numerical values for the dependent variable at the mesh points are obtained by replacing each derivative of the PDE at a mesh point by a finite difference approximation. These finite difference approximations are in terms of the dependent variable at the mesh point, neighbouring mesh points and also at the boundary points. From this we get a set of algebraic $eq^{n}s$. These $eq^{n}s$ can be solved by iteration.

The value of T at any point (i, j) can be obtained by using the eq^{n}, ~~$T(i,j) =$~~

$$T_{ij} = \frac{T_{i-1,j} + T_{i+1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$



This eqⁿ is known as standard five point formula.

To approximate the values at the grid points, when the boundary conditions are given, ~~th~~ we use the standard diagonal five point formula.

$$T_{ij} = \frac{T_{i-1,j+1} + T_{i+1,j+1} + T_{i-1,j-1} + T_{i+1,j-1}}{4}$$